ON EGOROFF'S THEOREM FOR NON-ADDITIVE MULTI MEASURES (Nonlinear Analysis and Convex Analysis)

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ON EGOROFF'S THEOREM FOR NON-ADDITIVE MULTI MEASURES

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ABSTRACT. Egoroff's theorem is established for set-valued measures, which take values in the family of all non-void, closed subsets of a real normed space using Hausdorff metric by several authors. In this paper, we prove Egoroff's theorem remains valid for non-additive measures, which take values in a family of sets of topological vector spaces using two types of convergency of set sequences.

1. INTRODUCTION

Egoroff's theorem is one of the most fundamental theorems in classical measure theory and does not necessary hold in non-additive measure theory without additional conditions. In [1], Wang generalized Egoroff's theorem in case of fuzzy measures, which are autocontinuous from above. Moreover in [2], Wang and Klir gave another generalization of this result for fuzzy measures, which are null-additive. In [3], Li showed that Egoroff's theorem remains true for fuzzy measures without any other supplementary conditions for them. When a fuzzy measure is not necessarily finite, Li et al. [4] have proved that Egoroff's theorem remains valid on fuzzy measures possessing the order continuity and pseudo-metric generating property. In [5], Murofushi, Uchino and Asahina find the necessary and sufficient condition called the Egoroff condition, which assures that Egoroff's theorem remains valid for real valued non-additive measures, see also Li [6] and Kawabe [7, 8] extend these results for Riesz space-valued fuzzy measures. Also these results for an ordered vector space-valued and an ordered topological vector space-valued non-additive measures, see [9, 10]. For information on real valued non-additive measures, see [2, 11, 12].

By several authors, Egoroff's theorem is established for non-additive multi measures, which take values in the family of all non-void, closed subsets of real normed spaces. In [13], Precupanu and Gavriluţ investigate Egoroff's theorem in a fuzzy multimeasure in the sense of Precupanu and et al. [14]. In [15], Wu and Liu investigate Egoroff's theorem in a set-valued fuzzy measure introduced by Gavriluţ [16].

In this paper, we prove Egoroff's theorem remains valid for non-additive multi measures. In particular, we use a topological convergence with respect to set-valued mappings, see [17, 18]. We consider Egoroff's theorem in set-valued situations and give two sufficient conditions of it. One is based on continuity from above and below, another is base on strongly order continuous and property (S) in set-valued cases. Next paper we give another sufficient condition to establishment of set-valued Egoroff's theorem.

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2. Preliminaries

Let R be the set of all real numbers and N the set of all natural numbers. We denote by $\mathcal{T}$ the set of all mappings from N into N. Let X be a non-empty set and $\mathcal{F}$ a σ-field of X. Let Y be a topological vector space (see [19, 20]). Let θ be an origin of Y, and $\mathcal{B}_\theta$ a system of neighborhoods of $\theta \in Y$. We denote $\mathcal{P}_0(Y)$ be a family of non-empty subsets of Y. Let $\mathcal{P}_{cl}(Y)$ be a family of closed, non-void subsets of Y. In this paper we consider the following two types convergence. Let $\{E_n\} \subset \mathcal{P}_0(Y)$ be a set sequence and $E \in \mathcal{P}_0(Y)$. We say that $\{E_n\}$ is

(A) type (I) convergent to E, if for any $e \in E$ there exists a sequence $\{e_n\}$, which converges to e, that is, for any $U \in \mathcal{B}_0$ there exists a n_0 with $e_n - e \in U$ for any $n \geq n_0$, such that $e_n \in E_n$ for every n;

(B) type (II) convergent to E, if given $j \in J$, for any sequence $\{e_{n_j}\} \subset Y$, which converges to $e \in Y$, that is, for any $U \in \mathcal{B}_0$ there exists a j_0 with $e_{n_j} - e \in U$ for any $j \geq j_0$, if $e_{n_j} \in E_{n_j}$, then $e \in E$.

If (A) holds, we will write $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty} E_n = E$ and if (B) holds, we will write $\mathrm{Lim}^{(\mathrm{II})}_{n\to\infty} E_n = E$. If both (A) and (B) hold, we will write $\mathrm{Lim}_{n\to\infty} E_n = E$ and said to be Kuratowski convergence [17, 18].

3. The continuity of non-additive multi measures

Definition 1. *Let $(X, \mathcal{F})$ be an arbitrary measurable space, and let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a set-valued mapping. μ is said to be a non-additive multi measure on X if the following conditions (i) and (ii) hold.*

(i) $\mu(\emptyset) = \{\theta\}$,

(ii) *for $A, B \in \mathcal{F}$ with $A \subset B$, $\mu(A) \subset \mu(B)$ (monotonicity).*

Moreover, we consider the following conditions.

Definition 2. *Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure. μ is said to be*

(i) *continuous from above type (I) if $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu(A_n) = \mu(A)$ whenever $\{A_n\} \subseteq \mathcal{F}$ and $A \in \mathcal{F}$ satisfy $A_n \searrow A$;*

(ii) *continuous from below type (I) if $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu(A_n) = \mu(A)$ whenever $\{A_n\} \subseteq \mathcal{F}$ and $A \in \mathcal{F}$ satisfy $A_n \nearrow A$;*

(iii) *continuous from above type (II) if $\mathrm{Lim}^{(\mathrm{II})}_{n\to\infty}\mu(A_n) = \mu(A)$ whenever $\{A_n\} \subseteq \mathcal{F}$ and $A \in \mathcal{F}$ satisfy $A_n \searrow A$;*

(iv) *continuous from below type (II) if $\mathrm{Lim}^{(\mathrm{II})}_{n\to\infty}\mu(A_n) = \mu(A)$ whenever $\{A_n\} \subseteq \mathcal{F}$ and $A \in \mathcal{F}$ satisfy $A_n \nearrow A$.*

Example 3. *Let $(X, \mathcal{F})$ be a measurable space, $m : \mathcal{F} \to R_+$ a non-additive measure on $\mathcal{F}$, $Y = R^2$ and R^2_+ is a positive cone. Consider the order interval with respect to R^2_+ defined by*

$$[a, b]_{R^2_+} := \{y \in R^2 \mid y \in (a + R^2_+) \cap (b - R^2_+)\},$$

where $a, b \in R^2$.

Define $\mu(A) := [(0, m(A)), (m(A), m(A))]_{R^2_+}$ for any $A \in \mathcal{F}$. Then μ is a non-additive multi measure on $\mathcal{F}$.

Definition 4. *Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure. μ is said to be*

(i) *strongly order continuous type (I), if it is continuous from above at measurable sets of measure zero, that is, for any* $\{A_n\} \subset \mathcal{F}$ *and* $A \in \mathcal{F}$ *satisfying* $A_n \searrow A$ *and* $\mu(A) = \{\theta\}$, *it holds that* $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu(A_n) = \{\theta\}$;

(ii) *strongly order semi-continuous type (I), if for any* $\{A_n\} \subset \mathcal{F}$ *and* $A \in \mathcal{F}$ *satisfying* $A_n \searrow A$ *and* $\mu(A) \ni \theta$, *it holds that* $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu(A_n) \ni \theta$.

Definition 5. *Let* $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ *be a non-additive multi measure.* μ *is said to be*

(i) *null-additive, if for any* $B \in \mathcal{F}$ *with* $\mu(B) = \{\theta\}$, *it holds that*

$$\mu(A \cup B) = \mu(A)$$

for any $A \in \mathcal{F}$;

(ii) *null-subtractive if for any* $B \in \mathcal{F}$ *with* $\mu(B) = \{\theta\}$, *it holds that*

$$\mu(A \setminus B) = \mu(A)$$

for any $A \in \mathcal{F}$.

Theorem 6. *Let* $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ *be a non-additive multi measure. Then the null-additivity of* μ *is equivalent to the null-subtractivity of it.*

Proof. (1) Suppose μ is null-additive. Let $E \in \mathcal{F}$ and $F \in \mathcal{F}$ with

$$\mu(F) = \{\theta\}.$$

By the monotonicity of μ, $\mu(E \cap F) = \{\theta\}$. Note that since $E = (E \setminus F) \cup (E \cap F)$ and the null-additivity of μ,

$$\mu(E) = \mu\left((E \setminus F) \cup (E \cap F)\right) = \mu(E \setminus F),$$

which implies that μ is null-subtractive.
(2) Suppose μ is null-subtractive, and E and F are defined as in (1). Then $\mu(F \setminus E) = \{\theta\}$. Note that since

$$E = E \cup (F \cap F^c) = (E \cup F) \setminus (F \setminus E),$$

and the null-subtractivity of μ,

$$\mu(E) = \mu((E \cup F) \setminus (F \setminus E)) = \mu(E \cup F),$$

which implies that μ is null-additive. □

4. Egoroff's Theorem

Definition 7. *Let* $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ *be a non-additive multi measure.*

(1) *A double sequence* $\{A_{m,n}\} \subset \mathcal{F}$ *is called a* μ*-regulator if it satisfies the following two conditions.*
  (D1) $A_{m,n} \supset A_{m,n'}$ *whenever* $n \le n'$.
  (D2) $\mu\left(\cup_{m=1}^{\infty} \cap_{n=1}^{\infty} A_{m,n}\right) = \{\theta\}$.

(2) μ *satisfies the weak-Egoroff condition if for any* μ*-regulator* $\{A_{m,n}\}$, *there exists a* $\tau \in T$ *such that* $\mu(\cup_{m=1}^{\infty} A_{m,\tau(m)}) \ni \theta$ *holds.*

(3) μ *satisfies the Egoroff condition if for any* μ*-regulator* $\{A_{m,n}\}$, *there exists a* $\tau \in T$ *such that* $\mu\left(\cup_{m=1}^{\infty} A_{m,\tau(m)}\right) = \{\theta\}$ *holds.*

It is easy to check that the following lemma holds.

Lemma 1. *Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure. μ satisfies the weak-Egoroff condition (resp. Egoroff condition) if (and only if), for any double sequence $\{A_{m,n}\} \subset \mathcal{F}$ satisfying* (D2) *in* Definition 7 *and the following* (D1$'$)*, it holds that there exists a $\tau \in T$ such that $\mu\left(\cup_{m=1}^{\infty} A_{m,\tau(m)}\right) \ni \theta$ (resp. $\mu\left(\cup_{m=1}^{\infty} A_{m,\tau(m)}\right) = \{\theta\}$).*

(D1$'$) *$A_{m,n} \supset A_{m',n'}$ whenever $m \geq m'$ and $n \leq n'$.*

Definition 8. *Let $(X, \mathcal{F}, \mu)$ be the non-additive multi measure space, f_n and $f \in \mathcal{F}$ for $n = 1, 2, \ldots$.*

(1) *$\{f_n\}$ is said to converge to f μ-almost everywhere on X, which is denoted by $f_n \overset{a.e.}{\to} f$, if there exists $A \in \mathcal{F}$ such that $\mu(A) = \{\theta\}$ and $\{f_n\}$ converges to f on $X \setminus A$.*
(2) *$\{f_n\}$ is said to converge to f μ-weak-almost uniformly on X, which is denoted by $f_n \overset{w-a.u.}{\to} f$, if there exists $\{A_\gamma \mid \gamma \in \Gamma\} \subset \mathcal{F}$ and there exists $\gamma \in \Gamma$ such that $\mu(A_\gamma) \ni \theta$ and $\{f_n\}$ converges to f uniformly on $X \setminus A_\gamma$.*
(3) *We say weak-Egoroff theorem holds if for μ if $\{f_n\}$ converges μ-weak-almost uniformly (μ-w-a.u.) to f whenever it converges μ-almost everywhere (μ-a.e.) to the same limit.*
(4) *$\{f_n\}$ is said to converge to f μ-almost uniformly on X, which is denoted by $f_n \overset{a.u.}{\to} f$, if there exists $\{A_\gamma \mid j \in \gamma\} \subset \mathcal{F}$ and there exists $\gamma \in \Gamma$ such that $\mu(A_\gamma) = \{\theta\}$ and $\{f_n\}$ converges to f uniformly on $X \setminus A_\gamma$.*
(5) *We say Egoroff theorem holds if for μ if $\{f_n\}$ converges μ-almost uniformly (μ-a.u.) to f whenever it converges μ-a.e. to the same limit.*

Under the above settings we have the following theorems.

Theorem 9. *Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure. If μ satisfies the weak-Egoroff condition, then the weak-Egoroff theorem holds for μ.*

Proof. Let $\{f_n\}$ be a sequence of $\mathcal{F}$-measurable real valued functions on X and f also such a function. Assume that $\{f_n\}$ converges μ-a.e. to f. For each $m, n \in N$, put

$$A_{m,n} = \bigcup_{j=n}^{\infty} \{x \in X \mid |f_j(x) - f(x)| \geq \frac{1}{m}\}.$$

It is easy to see that $\{A_{m,n}\}$ is a μ-regulator. By the assumption, there exists a $\tau \in \mathcal{T}$ such that $\mu\left(\cup_{m=1}^{\infty} A_{m,\tau(m)}\right) \ni \theta$. Note that $\mathcal{T}$ is upward directed by point wise partial ordering. Put $B_\tau = \cup_{m=1}^{\infty} A_{m,\tau(m)}$, then $\mu(B_\tau) \ni \theta$. Since $\{B_\tau \mid \tau \in \mathcal{T}\}$ is decreasing and by the monotonicity of μ, it is a similar way to prove of Egoroff's theorem for an additive measure, we have $f_n \to f$ uniformly on each set $X \setminus B_\tau$. $\square$

Theorem 10. *Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure. Then the following two conditions are equivalent.*

(1) *μ satisfies the Egoroff condition.*
(2) *The Egoroff theorem holds for μ.*

Proof. It is enough to prove only (2)$\to$(1): Let $\{A_{m,n}\}$ be a μ-regulator. By Lemma 1, we assume that $A_{m,n} \supset A_{m',n'}$ whenever $m \geq m'$ and $n \leq n'$. For each $n \in N$, put $f_n = \sup_{i \in N}\left((\frac{1}{i})\chi_{A_{i,n}}\right)$ where χ_B denotes the characteristic function of B. Then we have

$$A_{m,n} = \{x \in X | f_n(x) \geq \frac{1}{m}\} = \bigcup_{j=n}^{\infty}\{x \in X | f_j(x) \geq \frac{1}{m}\}$$

for all $m, n \in N$. By (D2), we have

$$\mu\left(\bigcup_{m=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}\{x \in X | f_j(x) \geq \frac{1}{m}\}\right) = \{\theta\}.$$

This implies that $\{f_n\}$ converges μ-a.e. to 0. By assumption, $\{f_n\}$ converges μ-almost uniformly to 0. Then there exists a decreasing net $\{B_\gamma \mid \gamma \in \Gamma\} \subset \mathcal{F}$ and there exists a $\gamma \in \Gamma$ such that $\mu(B_\gamma) = \{\theta\}$ and $\{f_n\}$ converges to 0 uniformly on each set $X \setminus B_\gamma$. Then we can find a $\tau \in T$ such that $\bigcap_{m=1}^{\infty}\left(X \setminus A_{m,\tau(m)}\right) \supset X \setminus B_\gamma$. Thus $\mu\left(\bigcup_{m=1}^{\infty} A_{m,\tau(m)}\right) \subset \mu(B_\gamma)$, so we have $\mu\left(\bigcup_{m=1}^{\infty} A_{m,\tau(m)}\right) = \{\theta\}$. □

5. Sufficient conditions for weak-Egoroff's Theorem

Next we give several sufficient conditions for the establishment of weak-Egoroff condition.

Theorem 11. *We assume that Y is locally convex spaces. Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure. If μ satisfies continuous from above type (I), continuous from below type (II), and null-additive, then the weak-Egoroff condition holds for μ.*

Proof. We divide proof in two steps.

(Step 1) For any $U \in \mathcal{B}_0$ and for any $k \in N$, there exists a $V_k \in \mathcal{B}_0$ such that $2^k V_k \subset U$. Let $\{A_{m,n}\}$ be a μ-regulator and put

$$D = \cup_{m=1}^{\infty} \cap_{n=1}^{\infty} A_{m,n}.$$

Then for any $m \in N$ and $(n_1, \ldots, n_m) \in N^m$,

$$A_{1,n} \cup D \searrow D, A_{1,n_1} \cup A_{2,n} \cup D \searrow A_{1,n_1} \cup D, \ldots,$$

and

$$\cup_{j=1}^{m} A_{j,n_j} \cup A_{m+1,n} \cup D \searrow \cup_{j=1}^{m} A_{j,n_j} \cup D$$

hold as $n \to \infty$. Since $\mu(D) = \{\theta\}$ and μ is continuous from above type (I), $\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu(A_{1,n} \cup D) = \mu(D)$, that is, there exists an $e_n^1 \in \mu(A_{1,n} \cup D)$ and for V_1, there exists an $n_1 \in N$ such that $e_n^1 \in V_1$ for any $n \geq n_1$. For n_1, $A_{1,n_1} \cup A_{2,n} \cup D \searrow A_{1,n_1} \cup D$ as $n \to \infty$,

$$\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu\left(A_{1,n_1} \cup A_{2,n} \cup D\right) = \mu\left(A_{1,n_1} \cup D\right),$$

that is, there exists an $e_n^2 \in \mu(A_{1,n_1} \cup A_{2,n} \cup D)$ and for V_2, there exists an $n_2 \in N$ such that $e_n^2 - e_{n_1}^1 \in V_2$ for any $n \geq n_2$, then $e_{n_2}^2 \in \{e_{n_1}^1\} + V_2 \subset V_1 + V_2$. Repeating the argument, since

$$\cup_{j=1}^{m-1} A_{j,n_j} \cup A_{m,n} \cup D \searrow \cup_{j=1}^{m-1} A_{j,n_j} \cup D \text{ as } n \to \infty,$$

$$\mathrm{Lim}^{(\mathrm{I})}_{n\to\infty}\mu\left(\cup_{j=1}^{m-1} A_{j,n_j} \cup A_{m,n} \cup D\right) = \mu\left(\cup_{j=1}^{m-1} A_{j,n_j} \cup D\right),$$

that is, for any $e_{n_{m-1}}^{m-1} \in \mu\left(\cup_{j=1}^{m-1} A_{j,n_j} \cup D\right)$ with

$$e_{n_{m-1}}^{m-1} \in V_1 + V_2 + \cdots + V_{m-1} \subset \frac{U}{2} + \frac{U}{2^2} + \cdots + \frac{U}{2^{m-1}},$$

there exists $e_n^m \in \mu(\cup_{j=1}^{m-1} A_{j,n_j} \cup A_{m,n} \cup D)$ such that for V_m there exists n_m with $e_{n_m}^m - e_{n_{m-1}}^{m-1} \in V_m$. Then we have $e_{n_m}^m \in \{e_{n_{m-1}}^{m-1}\} + V_m \subset \sum_{j=1}^m V_j$ for any m. Since the topology is locally convex, $\sum_{j=1}^m V_j \subset U$, thus we have $e_{n_m}^m \in U$.

(Step 2) Noting that μ is null additive, we have

$$\mu(\cup_{j=1}^m A_{j,n_j}) = \mu(\cup_{j=1}^m A_{j,n_j} \cup D).$$

Let $\tau \in \mathcal{T}$ satisfy $\tau(j) = n_j$ $(j = 1, 2, \ldots)$. Since

$$\cup_{j=1}^m A_{j,\tau(j)} \nearrow \cup_{j=1}^\infty A_{j,\tau(j)}$$

as $m \to \infty$ and μ is continuous from below type (II), we have

$$\mathrm{Lim}_{m\to\infty}^{(\mathrm{II})} \mu\left(\cup_{j=1}^m A_{j,\tau(j)}\right) = \mu(\cup_{j=1}^\infty A_{j,\tau(j)}).$$

Since $\{A_{m,n}\}$ is a μ-regulator, take a subsequence $e_{n_{m_i}}^{m_i} \in \mu(\cup_{j=1}^{m_i} A_{j,\tau(j)})$ from $\{e_{n_m}^m\}$, which obtained in (Step 1), then $e_{n_{m_i}}^{m_i} \in U$ and we have

$$\theta \in \mu(\cup_{j=1}^\infty A_{j,\tau(j)}).$$

Thus the assertion holds. □

Next we consider another sufficient condition.

Definition 12 ([22]). *A non-additive multi measure μ is said to have property (S), if for any $\{E_n\} \subset \mathcal{F}$, with $\mathrm{Lim}_{n\to\infty}^{(\mathrm{I})} \mu(E_n) = \{\theta\}$, there exists a subsequence $\{E_{n_i}\}$ of $\{E_n\}$ such that $\mu\left(\cap_{j=1}^\infty \cup_{i=j}^\infty E_{n_i}\right) \ni \theta$.*

Definition 13. *The double sequence $\{r_{m,n}\}$ of sets in $\mathcal{P}_{cl}(Y)$ is called a topological regulator if it satisfies the following two conditions.*

(1) *$r_{m,n} \supset r_{m,n+1}$ for any $m, n \in N$.*
(2) *For any $m \in N$, it holds that $\cap_{n=1}^\infty r_{m,n} \ni \theta$.*

Definition 14. *We say that $\mathcal{P}_{cl}(Y)$ has property (EP) if for any topological regulator $\{r_{m,n}\}$ in $\mathcal{P}_{cl}(Y)$, there exists a sequence $\{P_k\}$ of set in $\mathcal{P}_{cl}(Y)$ satisfying the following two conditions.*

(1) $\mathrm{Lim}_{k\to\infty}^{(\mathrm{I})} P_k = \{\theta\}$.
(2) *For any $k \in N$ and $m \in N$, there exists an $n_0(m, k) \in N$ such that $\{r_{m,n}\} \subset P_k$ for any $n \geq n_0(m, k)$.*

Theorem 15. *Let $\mu : \mathcal{F} \to \mathcal{P}_{cl}(Y)$ be a non-additive multi measure. We assume that μ is strongly order semi-continuous type (I) and satisfies property (S). We assume that $\mathcal{P}_{cl}(Y)$ has property (EP). Then μ satisfies the weak-Egoroff condition.*

Proof. Let $\{A_{m,n}\}$ be a μ-regulator. By Lemma 1, we are able to assume that $A_{m,n} \supset A_{m',n'}$ whenever $m \geq m'$ and $n \leq n'$. Then for any $m \in N$, $A_{m,n} \searrow \cap_{n=1}^\infty A_{m,n}$ and $\mu\left(\cap_{n=1}^\infty A_{m,n}\right) = \{\theta\}$ hold. By the monotonicity of μ, $\{\mu(A_{m,n})\}$ is a topological regulator in $\mathcal{P}_{cl}(Y)$. Since $\mathcal{P}_{cl}(Y)$ has property (EP), there exists a sequence $\{P_m\}$ of set such that $\cap_{m=1}^\infty P_m = \{\theta\}$ with the property that for any $m \in N$, there exists an $n_0(m) \in N$ such that $\mu(A_{m,n_0(m)}) \subset P_m$. So that $\mathrm{Lim}_{m\to\infty}^{(\mathrm{I})} \mu(A_{m,n_0(m)}) = \{\theta\}$. Since μ has property (S), there exists a strictly increasing sequence $\{m_i\} \subset N$ such that

$$\mu\left(\cap_{j=1}^\infty \cup_{i=j}^\infty A_{m_i,n_0(m_i)}\right) \ni \theta.$$

By the strongly order semi-continuity type (I) of μ, we have

$$\mathrm{Lim}^{(\mathrm{I})}_{j\to\infty}\mu(\cup_{i=j}^{\infty}A_{m_i,\,n_0(m_i)}) \ni \theta.$$

Thus there exists $a_j \in \mu\left(\cup_{i=j}^{\infty}A_{m_i,\,n_0(m_i)}\right)$ such that there exists j_0 and $a_j \in U$ for any $j \geq j_0$. Thus there exists a $j_0 \in N$ such that $\mu(\cup_{i=j_0}^{\infty}A_{m_i,\,n_0(m_i)}) \ni \theta$. Define $\tau \in \mathcal{T}$ such that $\tau(m) = n_0(m_{j_0})$ if $1 \leq m \leq m_{j_0}$ and $\tau(m) = n_0(m_i)$ if $m_{i-1} < m \leq m_i$ for some $i > j_0$. Since $\{A_{m,n}\}$ is increasing for each $n \in N$, it holds that

$$\cup_{i=j_0}^{\infty}A_{m_i,\,n_0(m_i)} = \cup_{m=1}^{\infty}A_{m,\,\tau(m)}.$$

Then μ satisfies the weak-Egoroff condition. □

Remark 16. *If we consider Hausdorff metric as the convergence of set-valued, then weak-Egoroff condition (resp. weak-Egoroff theorem) and Egoroff condition (resp. Egoroff theorem) are equivalent. Other conditions also would be redefined, see* [13, 23].

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